FINITE AMPLITUDE WAVES IN FLUID FILLED RUBBER

TUBES OF CIRCULAR CROSS SECTION

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Abstract

The dynamics of a fluid filled rubber tube of circular cross section are investigated using the Lagrangian plan of coordinates. The analysis is based on a one dimensional laminar flow model and includes large deformations. The adequacy of a one-dimensional model for studying blood flow in arteries has been experimentally and analytically investigated by Olsen and Shapiro who used the Eulerian plan of coordinates in their analysis.

The present investigation shows that the Lagrangian coordinates reduce the mathematical problem to the solution of the linear wave equation. This reduction is unique to the rubber tube with circular cross section. Therefore, the conclusions based on experiments and theory with rubber tubes of circular cross section should be carefully examined before they are applied to other elastic tubes or to arteries and veins. It is suggested that the experimental investigation of wave distortion (nonlinear wave equation) be investigated by using partially collapsed rubber tubes.

1. Introduction

The mathematical analysis of blood flow has generated many analyses of the dynamics of fluid filled elastic tubes.[1]* Although the majority of papers on the subject have been published within the last two decades, the problem was considered as early as 1882 by Leonhard Euler. The analytical treatment has been primarily based on small amplitude linear theories. Several authors have considered the non-linear terms by perturbation methods or by direct numerical integration of the characteristics.[2-7] Only recently have Olsen and Shapiro[8] (1967) attempted to establish a large amplitude theory which could be verified by experiments (on a simulated artery) in which the physical parameters could be accurately measured and controlled.

The analytical work of Olsen and Shapiro is based on a one dimensional flow model for a viscous incompressible liquid in a long elastic tube. In addition, it is assumed that (i) the wavelength is long compared with the tube diameter (i.e., the axial bending rigidity of the tube can be neglected), (ii) the tube is constrained from longitudinal motions, and (iii) the tube material follows the stress-strain law given by the kinetic theory for rubber. Two solutions were presented for the equations of motion. The first was a perturbation solution including second order terms, and the second solution was a direct numerical integration of the characteristics. In their experimental work, they considered standing waves in water filled rubber tubes. The agreement of the analytical model with laminar flow and the experiments was shown to be good over the physiological range of parameters.

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* Superscripts in text denote references.

The choice of rubber tubing for the simulation of arteries is quite logical since it is an elastic material which can withstand large strains. However, the unique pressure area relation of an axially constrained rubber tube of circular cross section (the model of Olsen and Shapiro) leads to conclusions regarding wave distortion which will not apply to a more general pressure area relation.

The purpose of this report is to show that the mathematical model for the Olsen-Shapiro problem can be obtained as a linear equation if the Lagrangian rather than the Eulerian coordinate system is used to formulate the problem. This not only enables one to obtain a closed form solution for the problem but shows immediately that the distortion of traveling waves is due only to the viscous forces and, accordingly, if viscous forces are small there will be small amounts of wave distortion. Conversely, if the pressure area relation for the tube is different from that of a rubber tube the wave distortion will occur in addition from other than viscous sources.

The results of this report do not prohibit the experimental study of non-linearities of the fluid flow (i.e., turbulence) which occur when the rubber tube has circular cross section. If the tube is of noncircular cross section (i.e., partially collapsed) then the effect of wave distortion due to the tube pressure area relation may be considered. This later problem has been considered by the author[9] in the study of the distortion of simple (nonreflecting) traveling waves.

2. Symbols

A   the internal cross sectional area of tube

a,b   constants defined by equation (21)

c   wave velocity

f   pressure area relation for tube

G   elastic constant for rubber

$g(x)$ function defined by equation (21)

h   tube wall thickness

L   length of tube

$\ell(x)$ function defined by equation (21)

M_{10} constant defined by equation (11)

p   pressure inside tube

Q_1 fow rate entering tube

R   tube internal radius

t   time

u   fluid particle displacement

$u_0(x)$ function defined by equation (19)

$\bar{u}_0$ constant defined by equation (18)

x   spatial coordinate

α constant defined by equation (12)

γ_R, γ_I constants defined by equation (22)

ϵ_{10} constant defined by equation (11)

ζ_1, ζ_2 constants defined by equation (15) and (16)

ν kinematic viscosity of fluid

ρ fluid density

τ wall shear force per unit length due to fluid viscosity

ω frequency of oscillatory flow

Subscripts

n   state of no pressure difference across tube wall

e   state of equilibrium with pressure difference across tube wall

3. Analysis

(a) Lagrangian Equations of Motion

Consider the equilibrium of an element of fluid which had original length δx (figure 1). Let $u(x,t)$ be the displacement of the particle which had coordinate x at time t=0. In addition, define

A_e - the original cross sectional area of the fluid element

$A(x,t)$ - the cross sectional area of the fluid element which had coordinate x at

             t=0 (i.e., the cross sectional area of the element at x + u).

$p(x,t)$ - the pressure in the fluid element which had coordinate x at t=0 (i.e., the

             pressure in the element at x + u).

ρ - fluid density (a constant for incompressible fluid)

τ - the wall shear force per unit length of tube.

The forces acting on the element surfaces are the pressure force

$$- \frac{\partial p}{\partial x} A \delta x$$

and the shear force from the tube wall

$$- \tau \left(1 + \frac{\partial u}{\partial x}\right) \delta x$$

These surface forces must balance the inertia force

$$\rho \delta x \, A_e \, \frac{\partial^2 u}{\partial t^2}$$

The equilibrium equation is

$$- \frac{\partial p}{\partial x} A - \tau \left(1 + \frac{\partial u}{\partial x}\right) = \rho A_e \frac{\partial^2 u}{\partial t^2} \tag{1}$$

The conservation of mass (or volume for an incompressible fluid) requires that

$$A_e = \left(1 + \frac{\partial u}{\partial x}\right) A \tag{2}$$

If the pressure area curve of the tube is given by

$$p = f(A) \tag{3}$$

then

$$\frac{\partial p}{\partial x} = \frac{df(A)}{dA} \frac{\partial A}{\partial x} = f'(A) \frac{\partial A}{\partial x} \tag{4}$$

Note that from differentiation of equation (2) with respect to x one obtains

$$\frac{\partial A}{\partial x} = \frac{- A_e \frac{\partial^2 u}{\partial x^2}}{\left(1 + \frac{\partial u}{\partial x}\right)^2} \tag{5}$$

Substituting from equations (4) and (5) into equation (1) leads to

$$f'(A) \frac{A^2}{A_e} \frac{\partial^2 u}{\partial x^2} = \tau \frac{A_e}{A^2} + \rho \frac{A_e}{A} \frac{\partial^2 u}{\partial t^2} \tag{6}$$

Equations (2) and (6) are the Lagrangian form of the equations of motion for u and A. These equations are equivalent to the Eulerian equations presented by

Olsen and Shapiro. The mathematical model applicability to physical problems which was demonstrated by their experiments is therefore, appropriate for equations (2) and (6).

For most pressure area curves and shear force relations, equations (2) and (6) lead to a nonlinear system of equations. Closed form solutions for these general equations are not available, however for simple (nonreflecting) waves and invisicid flow ($\tau = 0$) the solutions have been obtained by the author.[9]

(b) Rubber Tube Pressure Area Relation

The pressure area relation for a rubber tube of circular cross section is[8]

$$f(A) = G \frac{h_n}{R_n} \left[1 - \left(\frac{A_n}{A} \right)^2 \right] \tag{7}$$

where the quantities are defined

G - elastic constant which is a property of the rubber

h - tube wall thickness

R - tube internal radius

and the subscripts

n - the state of no differential pressure across the tube wall

e - the state of equilibrium at the time analysis begins (t=0)

Optionally, equation (7) may be written

$$f(A) = \frac{f_e}{1 - \left(\frac{A_n}{A_e} \right)^2} \left[1 - \left(\frac{A_n}{A} \right)^2 \right] \tag{8}$$

where

$$f_e = f(A_e) \tag{9}$$

or f_e is the equilibrium pressure when the analysis begins. Note that

$$f'(A) = \frac{2f_e}{\left(\dfrac{A_e}{A_n}\right)^2 - 1} \frac{A_e^2}{A^3} \tag{10}$$

(c) Wall Shear Force for Laminar Flow

The approximation for wall shear force is that resulting from a purely sinusoidal laminar flow in a rigid tube $Q = Q_0 e^{i\omega t}$, where Q is the flow and ω is the frequency of oscillation. With the assumption that the elastic tube condition can be approximated in a quasi-dynamic fashion by the rigid tube theory (the assumption of Olsen and Shapiro which was shown to be valid experimentally) the shear force becomes[2]

$$\tau = - i\omega\rho \, A\left\{1 - M_{10}^{-1} e^{-i\epsilon_{10}}\right\} \frac{\partial u}{\partial t} \tag{11}$$

M_{10} and ϵ_{10} are constants defined by

$$M_{10} e^{i\epsilon_{10}} = 1 - \frac{2J_1\left(\alpha i^{3/2}\right)}{\alpha i^{3/2} \, J_0\left(\alpha i^{3/2}\right)} \tag{12}$$

and

$$\alpha = \left(\frac{\omega R_e}{\nu}\right)^{1/2} \tag{13}$$

where ν is the kinematic viscosity and J is the Bessel function of the first kind. Tables of M_{10} and ϵ_{10} are given by Womersley.[2]

(d) Olsen-Shapiro Problem in Lagrangian Coordinates

The Lagrangian form of the problem considered by Olsen and Shapiro is obtained by introducing the pressure area relation for a rubber tube (equation

(10)) and the wall shear force for laminar flow (equation (11)) into the general

equation of motion (equation (6)). One obtains

$$c^2 \frac{\partial^2 u}{\partial x^2} = \omega \left[\zeta_1 + i\left(\zeta_2 - 1 \right) \right] \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2} \tag{14}$$

where

$$c^2 = \frac{2f_e}{\rho \left[\left(\frac{A_e}{A_n} \right)^2 - 1 \right]} \tag{15}$$

$$\zeta_1 = \frac{\sin \epsilon_{10}}{M_{10}} \tag{16}$$

$$\zeta_2 = \frac{\cos \epsilon_{10}}{M_{10}} \tag{17}$$

Several important conclusions can be drawn from the somewhat astonishing

result that the Lagrangian form of the equation of motion (equation (14)) is linear

for large as well as small deformations.

(a)   If the viscosity is zero ($\zeta_1 = 0, \zeta_2 = 1$) there will be no wave distortion.

This is a unique feature of the rubber tube (with circular cross section)

pressure area relation.

(b)   For viscous laminar flow the equation of motion is still linear and the

wave distortion is due to viscous effects (i.e., small wave distortion for

small viscosities). This is in agreement with the experimental results

of Olsen and Shapiro who concluded that "non-linear effects due to large

amplitude motion (in a rubber tube) are found to be not as large as those

in similar problems in gas dynamics and water waves."

(c) The mathematical problem in the Lagrangian coordinate system

(equation (14)) is identical to the first order perturbation problem of

the Eulerian system (presented by Olsen and Shapiro). That is, the

two problems have identical equations and coefficients.* This can be

rationalized on the basis that the Lagrangian and Eulerian equations

must be the same for small displacements. Since (for the case

considered here) the Lagrangian form is linear for large amplitudes,

it follows that the linearized Eulerian form must be the same as the

general Lagrangian form.

(d) The Lagrangian form of the equation of motion has an easily obtainable

closed form solution. This formulation is especially suited to the study of

standing waves. In contrast, the characteristics method of solution to

the Eulerian formulation requires the solution of an initial value

problem although the steady state solution is desired.

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* The problem for the first order perturbation (v_1) in Eulerian coordinates for

the velocity v is

$$c^2 \frac{\partial^2 v_1}{\partial x^2} = \omega \left[\zeta_1 + i\left(\zeta_2 - 1\right) \right] \frac{\partial v_1}{\partial t} + \frac{\partial^2 v_1}{\partial t^2}$$

In this equation $v(x,t)$ is the velocity at x at time t.

(e) Steady State Solution for Standing Waves

The wall shear force term is based on steady state sinusoidal flow and is valid only for that condition. The solution for standing waves is a tube plugged at one end, x=L, (no mean flow) and with prescribed sinusoidal displacement* of the fluid at the other end, x=0, requires the boundary conditions

$$u(L,t) = 0 \qquad (18)$$

$$u(0,t) = \bar{u}_0 e^{i\omega t} \qquad (19)$$

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* This boundary condition differs from that of the Olsen-Shapiro experiments in that they prescribed the flow at x=0 to be sinusoidal, i.e.

$$A \frac{\partial u}{\partial t} = Q_1 e^{i\omega t} \quad \text{at} \quad x = 0$$

This points out the difficulty of satisfying boundary conditions with the Lagrangian formulation. It is feasible, however to change the experimental apparatus to concur with the boundary condition (19). Or since the area of the tube near the inlet experienced only small changes in the Olsen-Shapiro experiments, the exact boundary condition can be approximated very accurately by

$$A_e \frac{\partial u}{\partial t} (0,t) = Q_1 e^{i\omega t}$$

or

$$\frac{\partial u}{\partial t} (0,t) = \frac{Q_1}{A_e} e^{i\omega t}$$

which is equivalent to boundary condition (18) with

$$\bar{u}_0 = - i \frac{Q_1}{A_e \omega}$$

The steady-state (standing wave) solution of equation (14) with boundary conditions (18) and (19) has the form

$$u(x,t) = u_0(x)e^{i\omega t} \tag{20}$$

Substitution of u(x,t) from equation (20) into equation (14) and retention of the function u (x) which satisfies the boundary conditions leads to the steady-state solution

$$u(x,t) = \frac{\bar{u}_0}{a - bi}\left[\ell(x) - ig(x)\right]e^{i\omega t} \tag{21}$$

where

$$
\begin{aligned}
a &= \sin \gamma_R L \cosh \gamma_I L \\
b &= \cos \gamma_R L \sinh \gamma_I L \\
\ell(x) &= \sin \gamma_R(L - x) \cosh \gamma_I(L - x) \\
g(x) &= \cos \gamma_R(L - x) \sinh \gamma_I(L - x)
\end{aligned} \tag{22}
$$

and

$$
\begin{aligned}
\gamma_R &= \frac{\omega}{c}\frac{1}{\sqrt{M_{10}}}\cos\left(\frac{\epsilon_{10}}{2}\right) \\
\gamma_I &= \frac{\omega}{c}\frac{1}{\sqrt{M_{10}}}\sin\left(\frac{\epsilon_{10}}{2}\right)
\end{aligned} \tag{23}
$$

If the boundary condition (equation (19)) is taken as the imaginary part of $\bar{u}_0 e^{i\omega t}$, that is,

$$u(0,t) = \text{Im}\left[\bar{u}_0 e^{i\omega t}\right] = \bar{u}_0 \sin \omega t \tag{24}$$

then

$$u(x,t) = \frac{\bar{u}_0}{a^2 + b^2}\left\{\left[a\,\ell(x) + b\,g(x)\right]\sin \omega t + \left[b\,\ell(x) - a\,g(x)\right]\cos \omega t\right\} \tag{25}$$

CONCLUDING REMARKS

The pressure area relation for a rubber tube leads to a linear equation of motion when the Lagrangian coordinates are used. The equation remains linear when an approximation for laminar flow wall shear stress is introduced into the analysis.

This unique feature of a rubber tube with circular cross section is fortunate from the mathematical standpoint, or if one is interested in determining the wave distortion due only to viscous effects. This feature is unfortunate if one is interested in studying the wave distortion due to the non-linearities of the pressure area relation.

An alternative to resorting to other materials in the study of wave distortion is to use partially collapsed (noncircular cross section) rubber tubes.

The conclusions of Olsen and Shapiro regarding the nonlinear effects of large amplitude motion in elastic tubes should be restricted to those elastic tubes which have the same pressure area relation as a rubber tube of circular cross section.

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Figure 1.- Coordinate system for Lagrangian equation development.

δx

x

A_e

$u(x+\delta x, t)$

$u(x, t)$

$A(x, t)$ $A(x+\delta x, t)$